6·9. FACTORIAL EXPERIMENTS

So far we have considered the testing of a number of treatments, not necessarily related to each other, in Randomised Blocks, or Latin Squares or Graeco Latin Squares. In industrial applications frequently we know that several factors may affect the characteristics in which we are interested, and we wish to estimate the effects of each of the factors and how the effect of one factor varies over the level of the other factors. For example, the yield of a chemical process may be affected by several factors such as the levels of pressure, temperature, rate of agitation, and proportions of reactants, etc. One might try to test each of the factors separately, holding all other factors constant in a given experiment, but with a little thought it might be clear that such an experiment might not give the information required. The logical procedure would be to vary all factors simultaneously, *i.e.*, within the framework of the same experiment. When we do so, we have what is now widely known as a *factorial experiment*.

2^3-experiment means an experiment with 3 factors at 2 levels each and 3^2-experiment means an experiment with 2 factors at 3 levels each.

Advantages of Factorial Experiment.

1. It increases the scope of the experiment and its inductive value and it does so mainly by giving information not only on the main factors but on their interactions.

2. The various levels of one factor constitute replications of other factors and increase the amount of information obtained on all factors.

3. When there are no interactions, the factorial design gives the maximum efficiency in the estimate of the effects.

4. When interactions exist, their nature being unknown a factorial design is necessary to avoid misleading conclusions.

5. In the factorial design the effect of a factor is estimated at several levels of other factors and the conclusions hold over a wide range of conditions.

Basic Ideas and Notations in the 2^n-Factorial Experiment. Let us first consider the design of the form 2^n in which there are n factors, each at two levels. Levels may be quite literally two quantitative levels or concentrations of, say, a fertilizer or it may mean two qualitative alternatives like two species of a plant. In some cases one level is simply the control group, *i.e.,* the absence of the factor and the other is its presence.

In order to develop extended notation to present the analysis of the design in a concise form, let us start, for simplicity with a 2^2-factorial design.

6·9·1. 2^2-Factorial Design. Here we have two factors each at two levels (0,1), say, so that there are $2 \times 2 = 4$ treatment combinations in all. Following the notations due to Yates, let the capital letters A and B indicate the names of the two factors under study and let the small letters a and b denote one of the two levels of each of the corresponding factors and this will be called the second level. The first level of A and B is generally expressed by the absence of the corresponding letter in the treatment combinations. The four treatment combinations can be enumerated as follows :

a_0b_0	or	'1'	:	Factors A and B, both at first level.
a_1b_0	or	a	:	A at second level and B at first level.
a_0b_1	or	b	:	A at first level and B at second level.
a_1b_1	or	ab	:	A and B both at second level.

These four treatment combinations can be compared by laying out the experiment in (*i*) R.B.D., with r replicates (say) , each replicate containing 4 units, or (*ii*) 4×4 L.S.D., and ANOVA can be carried out accordingly. In the above cases there are 3 *d.f.* associated with the *treatment effects*. In factorial experiment our main objective is to carry out separate tests for the main effects A, B and the interaction AB, splitting the treatment S.S. with 3 *d.f.* into three orthogonal components each with 1 *d.f.* and each associated either with the main effects A and B or the intersection AB.

Main Effects and Interactions. Suppose the factorial experiment with $2^2 = 4$ treatments is conducted in r-blocks or *replicates* as they are often called. Let [1], [a], [b] and [ab] denote the total yields of the r-units (plots) receiving the treatments 1, a, b and ab respectively and let the corresponding mean values obtained on dividing these totals by r be denoted by (1), (a), (b) and (ab) respectively. The letters A, B and AB when they refer to numbers will represent the main effects due to the factors A and B and their interaction AB respectively.

any factor ... present on the left.

Statistical Analysis of 2^2-design. Factorial experiments are conducted either in C.R.D. or R.B.D. or L.S.D. and thus they can be analysed in the usual manner except that in this case the treatment S.S. is split into three orthogonal components each with 1 *d.f.* It has already been pointed out that the main effects A and B, and the interaction AB are mutually orthogonal contrasts of treatment means. The S.S. due to the factorial effects A, B and AB is obtained by multiplying the squares of the factorial effects by a suitable quantity. In practice, these effects are usually computed from the treatment totals $[a]$, $[b]$, $[ab]$ etc., rather than from the treatment means (a), (b), etc. Factorial effect totals are given by the expressions :

$$\left.\begin{aligned} [A] &= [ab] - [b] + [a] - [1] \\ [B] &= [ab] + [b] - [a] - [1] \\ [AB] &= [ab] - [a] - [b] - [1] \end{aligned}\right\} \qquad \ldots (6\cdot221)$$

The S.S. due to any factorial effect is obtained on multiplying the square of the effect total by the factor $(1/4r)$, where r is the common replication number (*c.f.* Remark 7 below). Thus

$$\left.\begin{aligned} &\text{S.S. due to main effect of } A = [A]^2/4r \\ &\text{S.S. due to main effect of } B = [B]^2/4r \\ \text{and}\quad &\text{S.S. due to interaction } AB \ = [AB]^2/4r, \end{aligned}\right\} \qquad \ldots (6\cdot222)$$

each with 1 *d.f.*

TABLE 6·39 : ANOVA TABLE FOR FIXED EFFECT MODEL TWO FACTOR (2^2) EXPERIMENT IN R.B.D. IN 'r' REPLICATES

Source of Variation	*d.f.*	*S.S.*	*M.S.S.*	*Variance Ratio 'F'*
Blocks (Replicates)	$r-1$	S_R^2	$s_R^2 = \dfrac{S_R^2}{(r-1)}$	$F_R = s_R^2 / s_E^2$
Main effect A	1	$S_A^2 = [A]^2/4r$	$s_A^2 = S_A^2$	$F_A = s_A^2 / s_E^2$
Main effect B	1	$S_B^2 = [B]^2/4r$	$s_B^2 = S_B^2$	$F_B = s_B^2 / s_E^2$
Interaction $A \times B$	1	$S_{AB}^2 = [AB]^2 / 4r$	$s_{AB}^2 = S_{AB}^2$	$F_{AB} = s_{AB}^2 / s_E^2$
Error	$3(r-1)$	S_E^2 = By subtraction	$s_E^2 = S_E^2 / [3(r-1)]$	
Total	$4r-1$	$\sum_i \sum_j (y_{ij} - \bar{y}_{..})^2$		

Here each of the statistics F_A, F_B and F_{AB} follows central F-distribution with $[1, 3(r-1)]$ *d.f.* If for any factorial effect, calculated F is greater than tabulated F for $[1, 3(r-1)]$ *d.f.* and at certain level of significance 'say' α, then the null hypothesis H_0 of the presence of the factorial effect is rejected, otherwise H_0 may be accepted.

6·9·3. 2^3-Factorial Experiment. In 2^3-experiment we consider three factors, say, A, B, and C each at two levels, say, (a_0, a_1), (b_0, b_1) and (c_0, c_1) respectively, so that there are $2^3 = 8$ treatment combinations in all. Extending the notations due to Yates for a 2^2-experiment, let the corresponding small letters a, b and c denote the second level of each of the corresponding factors. The first level of each factor A, B and C is signified by the absence of the corresponding letter in the treatment combinations. The eight treatment combinations in a standard order are

$$\text{'1'}, a, b, ab, c, ac, bc, abc,$$

where, for example

$$1 = a_0 b_0 c_0, \quad a = a_1 b_0 c_0, \quad ab = a_1 b_1 c_0, \quad abc = a_1 b_1 c_1, \text{ etc.}$$

2^3-factorial experiment can be performed as a *C.R.D.* with 8 treatments, or *R.B.D.* with r replicates (say), each replicate containing 8 treatments of L.S.D. with $m = 8$ and data can be analysed accordingly. In 2^3-experiment we split up the treatment S.S. with 7 *d.f.* into 7 orthogonal components corresponding to the three main effects A, B and C, three first order (or two factor) interactions AB, AC, and BC and one second order interaction (or three factor interaction) ABC, each carrying 1 *d.f.* As in the case of 2^2-experiment A, B, AB, BC, etc., when they refer to numbers will represent the corresponding factorial effects.

interactions are important.

Main Effects and Interactions. Following the same notations for treatment totals and treatment means as in 2^2-factorial experiment, the simple effect of A, (say), is given by the differences in the mean yields of A as a result of increasing the factor A from the level a_0 to a_1, at other levels of the factors B and C.

Level of B	*Level of C*	*Simple effect of A*
b_0	c_0	$(a_1 b_0 c_0) - (a_0 b_0 c_0) = (a) - (1)$
b_1	c_0	$(a_1 b_1 c_0) - (a_0 b_1 c_0) = (ab) - (b)$
b_0	c_1	$(a_1 b_0 c_1) - (a_0 b_0 c_1) = (ac) - (c)$
b_1	c_1	$(a_1 b_1 c_1) - (a_0 b_1 c_1) = (abc) - (bc)$

... (6·226)

***Statistical Analysis of* 2^3-Design.** By using the *Table 6·44* of divisors and signs of a 2^3-factorial experiment, the various factorial effect totals can be expressed as mutually orthogonal contrasts of the 8 treatment totals. Thus, *e.g.*,

$$\left.\begin{aligned} [A] &= [abc] - [bc] + [ac] - [c] + [ab] - [b] + [a] - [1] \\ [AC] &= [abc] - [bc] + [ac] - [c] - [ab] + [b] - [a] + [1] \end{aligned}\right\} \quad \ldots(6{\cdot}235)$$

and so on. Another convenient way usually used for numerical computations of finding the factorial effect totals is the Yates' method as discussed in § 6·9·2.

In the analysis of 2^3-design we split the treatment S.S. with 7 *d.f.* into 7 mutually orthogonal components corresponding to seven factorial effects, each carrying 1 *d.f.* Obviously, the factorial effect totals are contrasts of the treatment totals and hence on using the result in (6·223*a*) [*c.f.* Remark 7 § 6·9·1], the S.S. due to any of the factorial effect is given by :

$$\frac{[\quad]^2}{\sum_{i=1}^{8} r.\,1} = \frac{[\quad]^2}{8r} \quad \ldots(6{\cdot}236)$$

i.e., S.S. due to any factorial effect, main or interaction is obtained on multiplying the square of the factorial effect total by $1/(8r)$, where r is the common replication number. Thus, for example

$$\left.\begin{aligned} \text{S.S. due to main effect } A &= \frac{[A]^2}{8r} \text{ with 1 } d.f., \\ \text{S.S. due to interaction } BC &= \frac{[BC]^2}{8r} \text{ with 1 } d.f., \end{aligned}\right\} \quad \ldots(6{\cdot}236a)$$

and so on.

ANOVA can now be carried out as given in the following table :

TABLE 6·45 : ANOVA TABLE FOR A 2^3-EXPERIMENT IN 'r' RANDOMISED BLOCKS

Source of Variation	*d.f.*	*S.S.*	*M.S.S.* $= \frac{S.S.}{d.f.}$	*Variance Ratio 'F'*
Replications (Blocks)	$(r-1)$	S_R^2	$s_R^2 = \frac{S_R^2}{r-1}$	$F_R = \frac{S_R^2}{S_E^2} \sim F[r-1, 7(r-1)]$
Main Effects				
A	1	$S_A^2 = [A]^2/8r$	$s_A^2 = S_A^2$	$F_A = s_A^2/s_E^2 \sim F[1, 7(r-1)]$
B	1	$S_B^2 = [B]^2/8r$	$s_B^2 = S_B^2$	$F_B = s_B^2/s_E^2 \sim F[1, 7(r-1)]$
C	1	$S_C^2 = [C]^2/8r$	$s_c^2 = S_C^2$	$F_C = s_C^2/s_E^2 \sim F[1, 7(r-1)]$
1st Order Interactions				
AB	1	$S_{AB}^2 = [AC]^2/8r$	$s_{AB}^2 = S_{AB}^2$	$F_{AB} = s_{AB}^2/s_E^2 \sim F[1, 7(r-1)]$
AC	1	$S_{AC}^2 = [AC]^2/8r$	$s_{AC}^2 = S_{AC}^2$	$F_{AC} = s_{AC}^2/s_E^2 \sim F[1, 7(r-1)]$
BC	1	$S_{BC}^2 = [BC]^2/8r$	$s_{BC}^2 = S_{BC}^2$	$F_{BC} = s_{BC}^2/s_E^2 \sim F[1, 7(r-1)]$
2nd Order Interaction				
ABC	1	$S_{ABC}^2 = [ABC]^2/8r$	$s_{ABC}^2 = S_{ABC}^2$	$F_{ABC} = s_{ABC}^2/s_E^2 \sim F[1, 7(r-1)]$
Error	$7(r-1)$	S_E^2 = By subtraction	$s_E^2 = \frac{S_E^2}{7(r-1)}$	
Total	$r.2^2 - 1 = 8r - 1$			

The hypothesis of the presence of a factorial effect is rejected at α% level of significance if the corresponding calculated F-statistic in the *Table 6·45* is greater than tabulated F_{α}; 1,7 $(r-1)$ otherwise the hypothesis may be accepted.

6·9·4. 2^n-Factorial Experiment. The results and the notations of 2^2 and 2^3 experiments can be generalised to the case of 2^n experiment. Here we consider n factors each at 2 levels. Suppose $A, B, C, D, \ldots, K$ are the factors each at two levels (0, 1). Corresponding small letters $a, b, c, d, \ldots, k$ denote the corresponding factors at the second level, the first level of any factor being signified by the absence of the corresponding small letter. The treatment combinations, in standard order, can be written as :

$$1, a, b, ab, c, ac, bc, abc, d, ad, bd, abd, cd, acd, bcd, abcd,$$
$$e, ae, be, abe, ce, ace, bce, abce, de, ade, bde, abde, cde, acde, bcde, abcde, \text{ etc.}$$

For 2^n-experiment, the various factorial effects are enumerated as follows :

Main effects	:	${}^nC_1$ in number
Two-factor interactions	:	${}^nC_2$ in number
Three-factor interactions	:	${}^nC_3$ in number
⋮		⋮
n factor interaction	:	${}^nC_n$ in number

Hence, the total number of factorial effects in 2^n-experiment are :

$$ {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = [{}^nC_0 + {}^nC_1 + \ldots + {}^nC_n] - 1 $$
$$ = (1+1)^n - 1 = 2^n - 1. \qquad \ldots (6\cdot239)$$

Main Effects and Interactions. As in the case of 2^2 and 2^3-experiments the results for the main effects and interactions can be generalised to the case 2^n-experiment. Thus, for n factors $A, B, C, D, \ldots, K$, the main effects and interactions are given by the expression :

$$\frac{1}{2^{n-1}}\Big[(a \pm 1)(b \pm 1)(c \pm 1)(d \pm 1)\ldots(k \pm 1)\Big] \qquad \ldots (6\cdot240)$$

the corresponding sign in each factor being taken as negative if the corresponding factor is contained in the factorial effect whose value we want. As usual, the R.H.S. is to be expanded algebraically and then the treatment combinations are to be replaced by the corresponding treatment means. The factorial effect totals can be obtained every conveniently from treatment totals by the generalisation of F. Yates' method as explained in § 6·9·2 for 2^2 and 2^3 experiments. As pointed out there, for 2^n experiment we shall need n cycles of the 'sum and difference' procedure.'

TABLE 6.46 : ANOVA TABLE FOR 2^n EXPERIMENT IN r RANDOMISED BLOCKS

Source of Variation	*d.f.*	*S.S.*	*M.S.S*
Blocks	$r-1$	$S_R^2 = \frac{\Sigma B_i^2}{2^n} - \text{C.F.}$	$s_R^2 = \frac{S_R^2}{r-1}$
Treatments	$2^n - 1$	$S_T^2 = \frac{\Sigma T_i^2}{r} - \text{C.F.}$	$s_T^2 = \frac{s_T^2}{2^n - 1}$
Main effects			
A	1	$S_A^2 = [A^2/r.\ 2^n$	$s_A^2 = S_A^2$
B	1	$S_B^2 = [B]^2/r.2^n$	$s_B^2 = S_B^2$
⋮	⋮	⋮	⋮
K	1	$S_K^2 = [K]^2 / r.2^n$	$s_K^2 = S_K^2$
Two-factor Interactions			
AB	1	$S_{AB}^2 = [AB]^2 / r.\ 2^n$	$s_{AB}^2 = S_{AB}^2$
AC	1	$S_{AC}^2 = [AC]^2 / r.\ 2^n$	$s_{AC}^2 = S_{AC}^2$
BC	1	$S_{BC}^2 = [BC]^2 / r.2^n$	$s_{BC}^2 = S_{BC}^2$
⋮	⋮	⋮	⋮
Three-factor Interactions			
ABC	1	$S_{ABC}^2 = [ABC]^2 / r.2^n$	$s_{ABC}^2 = S_{ABC}^2$
ACD	1	$S_{ACD}^2 = [ACD]^2 / r.2^n$	$s_{ACD}^2 = S_{ACD}^2$
⋮	⋮	⋮	⋮
n *-factor* interaction $ABCD...K$	1	$S_{AB...K}^2 = [AB...K]^2 / r.2^n$	$s_{AB...K}^2 = S_{AB...K}^2$
Error	$(r-1)(2^n-1)$	S_E^2 = By subtraction	$s_E^2 = \frac{S_E^2}{(r-1)(2^{n-1})}$
Total	$r.\ 2^{n-1}$	Raw S.S. – C.F.	

The block effects and the factorial effects (main and interactions) can be tested for significance by comparing their mean S.S. with error S.S.

6·9·2. Yates' Method of computing Factorial Effect Totals. For the calculation of various factorial effect totals for 2^n-factorial experiments ***F. Yates*** developed a special computational rule which enables us to avoid specific algebraic formulae, *e.g.*, the expressions in (6·221) for 2^2-factorial experiment. Yates' method consists in the following steps :

1. In the first column we write the treatment combinations. It is an essential part of the procedure that the treatment combinations be written in a standard systematic order as explained below :

"Starting with the treatment combination I, each factor is introduced in turn and is then followed by all combinations of itself with the treatment combinations previously written down, *e.g.*, for 2^2-experiment with factors A and B, the order of treatment combinations will be 1, a, b, ab and for 2^3 factorial experiment with factors A, B, and C, the order of treatment combinations will be 1, a, b, ab, c, ac, bc, abc, and so on. [For details of 2^3-experiment, notations, etc. see § 6·9·3.]

2. Against each treatment combination, write the corresponding total yields from all the replicates.

3. The entries in the third column can be split into two halves. The first half is obtained by writing down in order, the pairwise sums of the values in column 2 and the second half is obtained by writing in the same order the pairwise differences of the values in second column. It is to be remembered that the first member is to be subtracted from the second member of a pair.

4. To complete the next (4th) column, the whole of the procedure as explained in step 3 is repeated on column 3, and for 2^3-design, the 5th column is derived from 4th in a similar manner.

Thus for a 2^n-factorial experiment there will be n cycles of this "sum and difference" procedure. The first term in the last, *viz.*, $(n + 2)$th column always given the grand total (G) while the other entries in the last column are the totals of the main effects or the interactions corresponding to the treatment combinations in the first column of the table. In the *Tables 6·40 and 6·41*, we illustrate Yates' Method for 2^2 and 2^3 factorial experiments respectively.

TABLE 6·40 : YATES' METHOD FOR A 2^2-EXPERIMENT

Treatment Combination (1)	*Total Yield from all replicates* (2)	(3)	(4)	*Effect Totals*
'1'	[1]	[1] + [a]	[1] + [a] + [b] + [ab]	Grand Total
a	[a]	[b] + [ab]	[ab] – [b] + [a] – [1]	[A]
b	[b]	[a] – [1]	[ab] + [b] – [a] – [1]	[B]
ab	[ab]	[ab} – [b]	[ab] – [b] – [a] + [1]	[AB]

TABLE 6·41 : YATES' METHOD FOR A 2^3-EXPERIMENT

Treatment Combination (1)	*Treatment Totals* (2)	(3)	(4)	(5)	*Effect Totals*
'1'	[1]	[1] + [a] = u_1 (say)	$u_1 + u_2 = v_1$	$v_1 + v_2 = w_1$	Grand Total
a	[a]	[b] + [ab] = u_2 (say)	$u_3 + u_4 = v_2$	$v_3 + v_4 = w_2$	[A]
b	[b]	[c] + [ac] = u_3 (say)	$u_5 + u_6 = v_3$	$v_5 + v_6 = w_3$	[B]
ab	[ab]	[bc] + [abc] = u_4 (say)	$u_7 + u_8 = v_4$	$v_7 + v_8 = w_4$	[AB]
c	[c]	[a] – [1] = u_5 (say)	$u_2 - u_1 = v_5$	$v_2 - v_1 = w_5$	[C]
ac	[ac]	[ab] – [b] = u_6 (say)	$u_4 - u_3 = v_6$	$v_4 - v_3 = w_6$	[AC]
bc	[bc]	[ac] – [c] = u_7 (say)	$u_6 - u_5 = v_7$	$v_6 - v_5 = w_7$	[BC]
abc	[abc]	[abc] – [bc] = u_8 (say)	$u_8 - u_7 = v_8$	$v_8 - v_7 = w_8$	[ABC]

TABLE 6·42