Probability

5.1. INTRODUCTION TO PROBABILITY

Probability or chance is a word used generally in our day to day life. For example, a farmer say there is a chance of good yield of paddy than the last year, probably train may arrive within half an hour, there is 80% of chance to win the player A in the chess competition. These examples have some uncertainty and based on expectation by previous knowledge and analytical thinking. We need a quantitative measure of this uncertainty or expectations. For this purpose, the probability theory was introduced.

5.2. HISTORY OF PROBABILITY

The theory of probability was first introduced by Blaise Pascal and Pierre Fermat in 17th century related to the problems of gambling involving the dice. De-moivre and Pascal are two main contributors in the initial stage. James Bernoulli wrote "treatise on probability." De-moivre wrote "Doctrine of chances" in 1718. Some other contributors are Bayes, P.S. Laplace, Levy, Von Mises, R.A. Fisher, Chebychev, A. Markoff, Liapounoff, A. Khintchine and A. Kolmogrov in the development of various aspects of probability.

5.3. BASIC CONCEPTS OF PROBABILITY

1. Experiment

An experiment is a physical activity which has a several resultants known as outcomes.

Example. Tossing a coin

Measuring the length of the table.

Experiment is of two types : (*i*) Deterministic experiment; (*ii*) Undeterministic or random experiment.

2. Deterministic Experiment

If an experiment has certain resultant is called Deterministic Experiment.

Example. Measuring the Length of a table

Measuring the Width of the wall.

3. Undeterministic Experiment (or) Random Experiment

If an experiment is repeated under essential and homogeneous (identical) conditions and it has several possible outcomes, then the prediction of an outcome is not possible (*i.e.* outcome is not certain), is called a random experiment.

Example.
1. Tossing two coins
2. Throwing a die
3. Two players are playing a game.

4. Trial

A single performance of an experiment (*i.e.* conducting experiment once) is called a trial.

Example.
1. Tossing three coins
2. Playing pack of cards.

5. Outcome

The resultant of a random experiment is known as outcome.

Example.
1. In tossing a coin, head and tail are outcomes.
2. In throwing a die, 1, 2, 3, 4, 5, 6 points on the face of a die are outcomes.

6. Sample Space

The set of all possible outcomes in a random experiment is called a sample space. It is denoted by S.

Example :

1. In tossing two coins random experiment, the sample space S is
   S = {HH, HT, TH, TT}
2. In throwing a die random experiment, the sample space is
   S = {1, 2, 3, 4, 5, 6}
3. If three players A, B and C are playing a game, then sample space is
   S = {A, B, C}

7. Event

An event is a set of single or many outcomes of a random experiment. The event is a subset of sample space S. Event is denoted by E.

Example :

1. In tossing a coin, getting a head is an event.

2. In tossing two coins, getting atleast one head, *i.e.,* {HH, HT, TH} is an event.
3. In throwing two dice, getting same points on the two faces of the dice may be an event.

   E = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}.
4. In throwing two dice, getting the sum is 12 on the two faces of the dice may be treated as an event.

   E = {(6, 6)}

8. Elementary Event (or) Simple Event

A simple event or elementary event is an event which cannot be broken or divided further into smaller events.

Examples :

1. In tossing a coin, head and tail are simple events.
2. In throwing a die, getting the points 1, 2, 3, 4, 5, 6 are simple events.

9. Compound Event or Composite Event

Compound event is a combination of several simple events. Thus a compound event can be further divided into smaller events.

Example :

1. In throwing a die, getting an odd number is a compound event *i.e.* E = {1, 3, 5}
2. In tossing three coins, getting a head *i.e.* E = {HTT, THT, TTH} is a compound event.

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10. Exhaustive Events $(A \cup B) = S$

The total number of possible outcomes in a random experiment is known as exhaustive events.

Example :

1. In tossing a coin, head and tail are exhaustive events.
2. In throwing a die, 1, 2, 3, 4, 5, 6 are exhaustive events.

11. Mutually Exclusive Events $(A \cap B) = \phi$

No two or more events can occur in the same trial are called mutually exclusive events

(or)

Events are said to be mutually exclusive if the happening of any one of them precludes the happening of all the other events.

Example :

1. In tossing a coin, head and tail are mutually exclusive events.
2. In throwing a die, getting 1, 2, 3, 4, 5, 6 are mutually exclusive events.

10. Equally likely Events

The outcomes of a random experiment are said to be equally likely events if taking into consideration all relevant evidences, there is no reason to expect one in preference to the others.

Example :

1. In tossing a coin, getting a head or a tail are equally likely events.
2. In throwing a die, getting 1 to 6 are equally likely events.

11. Favourable Outcomes

The number of outcomes favourable to an event is the number of outcomes which entail the happening of the event.

Example :

1. In tossing two coins, for getting atleast one head event, favourable outcomes are {HT, TH, HH}
2. In throwing two dice, for the event of getting sum is 4, the favourable outcomes are (1, 3), (2, 2) (3, 1)

12. Independent Events

Two or more events are said to be independent if the happening or non-happening of one event is not affected by the happening or non-happening of the other events.

Example :

1. In tossing a coin twice, getting a head in the first trial is independent of getting a head in the second trial.
2. In drawing two cards from a pack of cards, drawing a card in the second draw is independent of drawing a card in the first draw if the selected card in the first draw was replaced before the second draw. Otherwise they are dependent *i.e.* if the card is not replaced before the second draw.

13. Complementary Event

If E is an event, then its complementary event consist outcomes of the sample space S and should not contain outcomes of E.

It is denoted by $\overline{E}$ or E^c or E^1.

$\overline{E} = S - E$

Example :

1. In tossing a coin, if event E is considered as getting a head then its complementary event is $\overline{E}$ = getting a tail.
2. In throwing a die, if event E is taken as getting odd number in the face of die, then $\overline{E}$ = getting even number = {2, 4, 6}.

5.4. DEFINITIONS OF PROBABILITY

1. Mathematical (or) Classical Definition of Probability

If there are '*n*' exhaustive, mutually exclusive and equally likely outcomes and '*m*' of them are favourable outcomes to the happening of an event E. Then the probability of happening the event E is

$$P(E) = p = \frac{\text{favourable number of outcomes}}{\text{exhaustive number of outcomes}} = \frac{m}{n}$$

2. Statistical or Empirical Definition of Probability

If a random experiment is repeated number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event happens to the number of trials when number of trials become indefinitely large, is called statistical or empirical probability. This is defined by Von Mises.

Symbolically, if event E happens 'm' times out of total n trials, then

$$P(E) = p = \lim_{n \to \infty} \frac{m}{n}$$

3. Probability Space

The triplet (S, B, P) is called probability space.

where S : Sample space
B : Borel σ-field is set of all subsets of S (*i.e.* contains all events)
P : Probability function defined on Borel σ-field.

4. Axiomatic Definition of Probability

Let (S, B, P) be a probability space. A function P defined on σ-field B satisfying the following axioms

(*i*) $P(E_i) > 0, \quad \forall\, i$ (positivity)

(*ii*) $P(S) = 1$ (certainty)

(*iii*) If the event $E_1, E_2, \ldots, E_n\ (\in B)$ are disjoint events, then

$$P(E_1 \cup E2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$$

i.e. $$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) \quad \text{(additivity)}$$

The probability P satisfying the axioms positivity, certainty and additivity is called axiomatic definition of probability.

Properties of Probability

1. For any event A, $P(A) \geq 0$.
2. For the impossible event, $P(\phi) = 0$.
3. The probability of the sample space (or) certain event is unity, *i.e.*, $P(S) = 1$.
4. For any event A, Probability of the complementary event of A is $P(\overline{A}) = 1 - P(A)$.
5. If A and B are mutually disjoint events, then
$$P(A \cup B) = P(A) + P(B)$$
6. For any two events A and B,
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

5.5. CONDITIONAL PROBABILITY

If A and B are any two events, then happening of event B after the event A has already happened is called the conditional event of B given A and is denoted by $B|A$. Happening of event A when the event B has already happened is known as conditional event of A given B and is denoted by $A|B$.

Definition. The probability of happening the event A when the event B has already happened is called conditional probability of A given B. It is denoted by P(A|B) and is given by

$$P(A|B) = \frac{n(A \cap B)}{n(B)}$$

(or)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) > 0. \qquad \text{[For A|B, the sample space is B]}$$

The probability of happening the event B when the event A has already happened is called the conditional probability of B given A. It is denoted by P(B|A) and is given by

$$P(B|A) = \frac{n(A \cap B)}{n(A)}$$

(or)

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, P(A) > 0. \qquad \text{[For B|A, the sample space is A]}$$

5.6. INDEPENDENT EVENTS

Two events A and B are said to be independent if and only if

$$P(A \cap B) = P(A) \,.\, P(B).$$

(or)

An event B is said to be independent of event A, if the conditional probability of B given A is equal to the unconditional probability of B.

i.e. $P(B|A) = P(B)$

$\therefore$ $P(A \cap B) = P(A) \,.\, P(B|A) = P(A) \,.\, P(B)$

|||ly $P(A|B) = P(A).$

and $P(A \cap B) = P(B)\, P(A|B) = P(B) \,.\, P(A).$

PROBLEMS

PROBLEM 1. If two coins are tossed, find the probability of getting atleast one head.

SOLUTION. If two coins are tossed, the sample space

$S = \{HH, HT, TH, TT\}, n = 4$

Let event E : getting atleast one head.

$\bar{E}$: getting no head

For $\bar{E}$, favourable outcomes = {TT} $\therefore$ $m = 1$

$\therefore$ $P(\bar{E}) = \frac{m}{n} = \frac{1}{4}$

$\therefore$ $P(E) = 1 - \frac{1}{4} = \frac{3}{4}.$

PROBLEM 2. If two dice are thrown, find the probability of getting a sum is 10.

SOLUTION. The sample space S = {(1, 1) (1, 2), ..., (6, 5), (6, 6)}

$$n = 6^2 = 36.$$

Let E : getting the sum is 10

$$E = \{(4, 6), (5, 5), (6, 4)\}$$

$$\therefore \; m = 3$$

$$p = \frac{3}{36}.$$

PROBLEM 3. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and red.

SOLUTION. Total number of outcomes = $^{16}c_2 = 120$

Let E be two drawn balls are white and red.

$$\therefore \quad m = {}^6c_1 . {}^3c_1 = 18$$

$$\therefore \quad p = \frac{m}{n} = \frac{18}{120} = \frac{3}{20}$$

PROBLEM 4. Two cards are drawn from a well shuffled pack of 52 cards. Find the probability that they are (*i*) two aces, (*ii*) a king and a queen.

SOLUTION. Total number of outcomes = $n = {}^{52}c_2 = 1326$

(*i*) Let event E be drawing two cards are aces

Favourable number of outcomes of E are

$$m = {}^4c_2 = 6$$

$$\therefore \quad P(E) = \frac{6}{1326} = \frac{1}{221}$$

(*ii*) Let event E be drawing cards are king and queen

Favourable outcomes for E are

$$m = {}^4c_1 . {}^4c_1 = 16$$

$$\therefore \quad P(E) = \frac{16}{1326} = \frac{8}{663}$$

PROBLEM 5. What is the probability that a leap year contains 53 Sundays.

SOLUTION. A leap year have 366 days, 52 weeks and two days leftover. The total number of outcomes are

∴ S = {(Sun, Mon), (Mon, Tue), (Tue, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat), (Sat, Sun)}

$\therefore \; n = 7$

52 week contains 52 Sundays and another Sunday required to get 53 Sundays.

Let E be 53 Sundays in a leap year.

Favourable outcomes of E are

$$E = \{(Sun, Mon), (Sat, Sun)\}$$

$$\therefore \quad m = 2$$

$$\therefore \quad P(E) = \frac{2}{7}$$

PROBLEM 6. Five digit numbers are formed with 0, 1, 2, 3, 4. Find the probability of getting 2 in the ten's place and 0 in the units place always.

SOLUTION. Total number of 5 digit numbers formed by using 0, 1, 2, 3, 4 are

$$n = 5! - 4! \quad \text{(or)} \quad 4 \times 4! = 96$$

A number starts with 0, it cannot be treated as five digit number, therefore 4 ! should be subtracted.

Let event E be getting the number with 2 in 10's place and 0 in units place always.

$$\therefore \quad m = 3!\,.\,1\,.\,1 = 6$$

$$\therefore \quad P(E) = \frac{m}{n} = \frac{6}{96} = \frac{1}{16}$$

PROBLEM 7. There are 4 letters and 4 envelopes and if all the letters are placed in the envelopes at random, then find the probability that atleast one letter will be place in wrong addressed envelop.

SOLUTION. 4 letters are placed in 4 envelops in 4 ! ways

$$n = 4! = 24$$

Let E be atleast one letter will be placed wrongly.

$\bar{E}$ is no letter will be placed wrongly (*i.e.* all are in correct addresses)

Favourable outcomes for $\bar{E}$

$$m = 1$$

$$P(\bar{E}) = \frac{1}{24}$$

$$P(E) = 1 - P(\bar{E}) = 1 - \frac{1}{24} = \frac{23}{24}.$$

PROBLEM 8. What is the probability that four S's come consecutively in the word MISSISSIPPI.

SOLUTION. MISSISSIPPI has 4S's 2P's, 4I's, 1-M.

Total number of outcomes

$$n = \frac{11!}{4!\,4!\,2!} = 34650$$

Let E be the event of getting four S's come consecutively.

Favourable outcomes of E are

$$m = \frac{8!}{4!\,2!} = 840$$

(Since if all 4 S's consider as one letter, then total 8 letters can be arranged in 8 ! ways)

$$\therefore \quad P(E) = \frac{m}{n} = \frac{840}{34650} = \frac{4}{165}$$

Theorems of Probability

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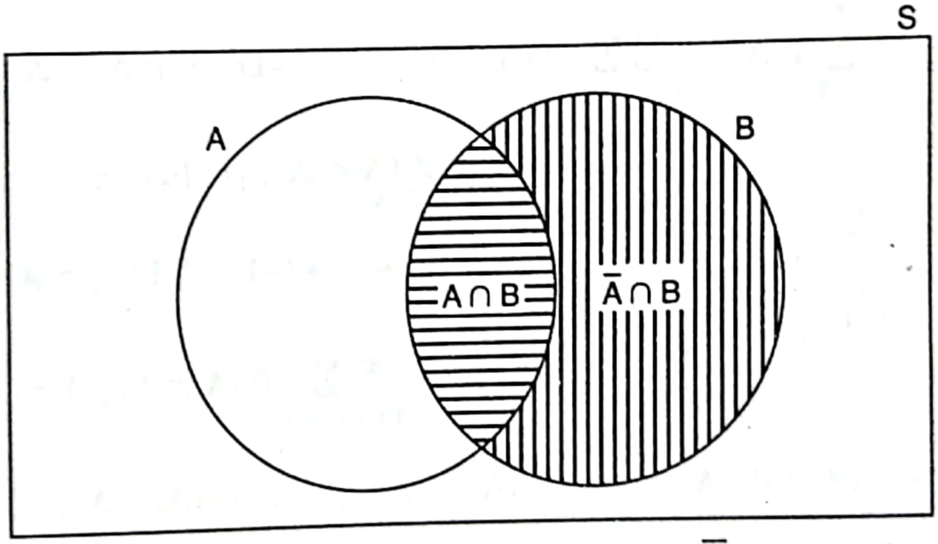
6.1. ADDITION THEOREMS

Addition Theorem of Probability for two events

Statement : If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof :



Express $A \cup B$ as union of two disjoint events A and $\bar{A} \cap B$ as shown in the Venn diagram.

$$A \cup B = A \cup (\bar{A} \cap B)$$

Consider probabilities,

$$P(A \cup B) = P(A \cup (\bar{A} \cap B))$$

$$= P(A) + P(\bar{A} \cap B) \qquad [\because \text{ from additivity}]$$

Add and subtract $P(A \cap B)$

$$\therefore \quad P(A \cup B) = P(A) + P(\overline{A} \cap B) + P(A \cap B) - P(A \cap B)$$

$$= P(A) + P[(\overline{A} \cap B) \cup (A \cap B)] - P(A \cap B)$$

[$\because$ $\overline{A} \cap B$ and $A \cap B$ are disjoint events]

$$= P(A) + P(B) - P(A \cap B)$$

[from Venn diagram]

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

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6.3. MULTIPLICATION THEOREMS

Multiplication Theorem of Probability for two events

Statement : For two events A and B,

$$P(A \cap B) = P(A), P(B|A), P(A) > 0$$

$$= P(B), P(A|B), P(B) > 0$$

where $P(B|A)$ is conditional probability of B given A, $P(A|B)$ is conditional probability of A given B.

Proof : Let A and B are two events in the sample space S, and let $n\,(A)$, $n\,(B)$ are elements (or) outcomes favourable to the events A and B respectively. Assume $n\,(A \cap B)$ be the favourable outcomes of $A \cap B$, $n\,(S)$ be the total number of outcomes in the sample space. Then by the definition of probability,

$$P(A) = \frac{n\,(A)}{n\,(S)}, \quad P(B) = \frac{n\,(B)}{n\,(S)}, \quad P(A \cap B) = \frac{n\,(A \cap B)}{n\,(S)}$$

$$P(A|B) = \frac{n\,(A \cap B)}{n\,(B)}, \quad P(B|A) = \frac{n\,(A \cap B)}{n\,(A)}$$

Consider

$$P(A \cap B) = \frac{n\,(A \cap B)}{n\,(S)}$$

Multiply and divide with $n\,(B)$

$$\therefore \qquad P(A \cap B) = \frac{n\,(A \cap B)}{n\,(S)} \cdot \frac{n\,(B)}{n\,(B)} = \frac{n\,(B)}{n\,(S)} \cdot \frac{n\,(A \cap B)}{n\,(B)}$$

$$= P(B), P(A|B), P(B) > 0$$

Now consider

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)}$$

Multiply and divide with $n(A)$

$$\therefore \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} \cdot \frac{n(A)}{n(A)} = \frac{n(A)}{n(S)} \cdot \frac{n(A \cap B)}{n(A)}$$

$$= P(A), P(B|A), \ P(A) > 0$$

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6.4. BAYES' THEOREM

Baye's Theorem. If $E_1, E_2, \ldots E_n$ are 'n' mutually disjoint events with $P(E_i) \neq 0$, $i = 1, 2, \ldots n$, then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have

$$P(E_i|A) = \frac{P(E_i) \,.\, P(A|E_i)}{\sum_{i=1}^{n} P(E_i) \,.\, P(A|E_i)}, \ i = 1, 2, \ldots, n.$$

Proof : Since $A \subset \bigcup_{i=1}^{n} E_i$, then we know

$$A = A \cap \bigcup_{i=1}^{n} E_i$$

$$A = A \cap (E_1 \cup E_2 \cup \ldots \cup E_n) = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$

$$= \bigcup_{i=1}^{n} (A \cap E_i)$$

Since $E_1, E_2, \ldots, E_n$ are mutually disjoint events, $A \cap E_1$, $A \cap E_2$, … $A \cap E_n$ are also disjoint events.

$\therefore$ By using additivity axiom of probability,

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right]$$

$$= P\,[(A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)]$$

$$= P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n) \qquad [\because \text{ from additivity}]$$

$$= \sum_{i=1}^{n} P(A \cap E_i)$$

$$P(A) = \sum_{i=1}^{n} P(E_i) \,.\, P(A|E_i) \qquad \ldots(1)$$

[This is called total probability]

Also we know that

$$P(A \cap E_i) = P(A) \,.\, P(E_i|A)$$

$$P(E_i|A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i) \,.\, P(A|E_i)}{\sum_{i=1}^{n} P(E_i) \,.\, P(A|E_i)}, \ i = 1, 2, \ldots n \qquad [\text{From } (1)]$$

6.2. BOOLE'S INEQUALITY

(*i*) **Statement :** For n events $A_1, A_2, \ldots A_n$, we have

$$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1).$$

Proof : This theorem can be proved by using mathematical induction.

For $n = 2$, we have

$$P(A_1 \cup A_2) \leq 1$$

$$P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$$

$$P(A_1) + P(A_2) - 1 \leq P(A_1 \cap A_2)$$

$$P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1 \qquad \ldots(1)$$

Hence the theorem is true for $n = 2$

Let us suppose that the theorem is true for $n = r$

$$P\left(\bigcap_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r} P(A_i) - (r-1) \qquad \ldots(2)$$

Now, we have to prove it for $n = r + 1$

Consider

$$P\left(\bigcap_{i=1}^{r+1} A_i\right) = P\left(\bigcap_{i=1}^{r} A_i \cap A_{r+1}\right) = P\left(\bigcap_{i=1}^{r} A_i\right) + P(A_{r+1}) - 1 \qquad \text{[from (1)]}$$

$$\geq \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1 \qquad \text{[from (2)]}$$

$$P\left(\bigcap_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - (r+1-1)$$

∴ The theorem is true for $n = r + 1$.

Hence by mathematical induction, the theorem is true for all values of n.

(*ii*) **Statement :** For n events $A_1, A_2, \ldots A_n$, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

Proof : We can prove this theorem by using mathematical induction.

For $n = 2$, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2) \qquad \ldots(1) \ [\because P(A_1 \cap A_2) \geq 0]$$

Hence the theorem is true for $n = 2$.

Let us suppose that the theorem is true for $n = r$.

$$P\left(\bigcup_{i=1}^{r} A_i\right) \leq \sum_{i=1}^{r} P(A_i) \qquad \ldots(2)$$

Now, we have to prove it for $n = r + 1$.

Consider

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left(\bigcup_{i=1}^{r} A_i \cup A_{r+1}\right)$$

$$\leq P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) \qquad \text{[from (1)]}$$

$$\leq \sum_{i=1}^{r} P(A_i) + P(A_{r+1}) \qquad \text{[from (2)]}$$

$$\leq \sum_{i=1}^{r+1} P(A_i)$$

Hence the theorem is true for $n = r + 1$.

$\therefore$ The theorem is true for all the values of n.