

Bivariate Random Variables

Bivariate Random Variables

(AU 2018)

Definition. Let X and Y be two random variables defined on the sample space S , then the random vector (X, Y) assigns a point in the two dimensional space R^2 is called two dimensional random variables or bivariate random variables.

Examples :

1. Heights and weights of a group of persons.
2. Income and expenditure of a group of families.
3. Volume and pressure of a gas.

Bivariate Distribution

Bivariate distribution can be defined for two kinds of random variables, discrete and continuous. There are three concepts—joint, marginals and conditionals for both discrete and continuous random variables separately.

2.1. BIVARIATE DISTRIBUTION : DISCRETE RANDOM VARIABLES

The joint, marginals and conditionals for discrete random variables can be explained below.

1. Joint Probability Mass Function

(AU 2017)

Let X and Y be two discrete random variables taking the values x_1, x_2, \dots, x_m and y_1, y_2, \dots, y_n respectively. The function p_{ij} defined on $X = x_i$ and $Y = y_j$ is called joint probability mass function of (X, Y) , is denoted by $P(X = x_i, Y = y_j)$ or $P(X = x_i \cap Y = y_j)$ or $P(x_i, y_j)$ and is represented in the following bivariate table.

Y \ X	X						Marginals of Y
	x_1	x_2	...	x_i	...	x_m	
y_1	P_{11}	P_{21}	...	P_{i1}	...	P_{m1}	$P_{\cdot 1}$
y_2	P_{12}	P_{22}	...	P_{i2}	...	P_{m2}	$P_{\cdot 2}$
\vdots	\vdots	\vdots		\vdots		\vdots	\vdots
y_j	P_{1j}	P_{2j}	...	P_{ij}	...	P_{mj}	$P_{\cdot j}$
\vdots	\vdots	\vdots		\vdots		\vdots	\vdots
y_n	P_{1n}	P_{2n}	...	P_{in}	...	P_{mn}	$P_{\cdot n}$
Marginals of X	$P_{1\cdot}$	$P_{2\cdot}$...	$P_{i\cdot}$...	$P_{m\cdot}$	$P_{\cdot\cdot} = 1$

Definition. The joint probability mass function of X and Y is denoted by $P(x_i, y_j)$ or p_{ij} or $P(X = x_i, Y = y_j)$ and it should satisfy the following conditions.

$$(i) P(x_i, y_j) \geq 0 \quad \forall (i, j)$$

$$(ii) \sum_{i=1}^m \sum_{j=1}^n P(x_i, y_j) = 1$$

2. Marginal Probability Functions

(AU 2011)

There are two marginal probability functions.

1. Marginal Probability Function of X

Marginal probability function of X is denoted by $P_X(x)$ or $P(X = x_i)$ or $p_{i\cdot}$ and defined as

$$\begin{aligned} P(X = x_i) &= P(X = x_i, Y = y_1) + P(X = x_i, Y = y_2) + \dots \\ &\quad \dots + P(X = x_i, Y = y_j) + \dots + P(X = x_i, Y = y_n) \\ &= p_{i1} + p_{i2} + \dots + p_{ij} + \dots + p_{in} \\ &= \sum_{j=1}^n p_{ij} \text{ (or) } \sum_{j=1}^n P(x_i, y_j), \quad i = 1, 2, \dots, m. \end{aligned}$$

Marginal probability function of X should satisfy the following conditions.

$$(i) p_{i\cdot} = P(X = x_i) \geq 0 \quad \forall i$$

$$(ii) \sum_{i=1}^m p_{i\cdot} = \sum_{i=1}^m P(X = x_i) = 1.$$

2. Marginal Probability Function of Y

Marginal probability function of Y is denoted by $P_Y(y)$ or $P(Y = y_j)$ or $p_{\cdot j}$ and defined as

$$\begin{aligned} P(Y = y_j) &= P(X = x_1, Y = y_j) + P(X = x_2, Y = y_j) + \dots \\ &\quad \dots + P(X = x_i, Y = y_j) + \dots + P(X = x_m, Y = y_j) \end{aligned}$$

$$= p_{1j} + p_{2j} + \dots + p_{ij} + \dots + p_{mj}$$

$$= \sum_{i=1}^m p_{ij} \text{ (or) } \sum_{i=1}^m P(X = x_i, Y = y_j) \text{ (or) } \sum_{i=1}^m P(x_i, y_j), j = 1, 2, \dots, n.$$

Properties of marginal probability function of Y

(i) $p_{\cdot j} = P(Y = y_j) \geq 0 \quad \forall j$

(ii) $\sum_{j=1}^n p_{\cdot j} = \sum_{j=1}^n P(Y = y_j) = 1$

3. Conditional Probability Functions

There are two conditional probability functions.

1. Conditional Probability Function of Y given X

The conditional probability function of Y given X is denoted by $P(Y = y_j | X = x_i)$ or

$P(y | x)$ and is defined as $P(Y = y_j | X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)}$

Properties of conditional probability function of Y given X :

(i) $P(Y = y_j | X = x_i) \geq 0 \quad \forall j$

(ii) $\sum_{j=1}^n P(Y = y_j | X = x_i) = 1$

2. Conditional Probability Function of X given Y

The conditional probability function of X given y is denoted by $P(X = x_i | Y = y_j)$ or

$P(x | y)$ and is defined as $P(X = x_i | Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$

Properties of conditional probability function of X given Y :

(i) $P(X = x_i | Y = y_j) \geq 0 \quad \forall i$

(ii) $\sum_{i=1}^m P(X = x_i | Y = y_j) = 1.$

Note : The notation of joint marginal and conditionals can also be represented simply as follows.

Joint : $P(x, y)$, Marginals : $P(x), P(y)$,

Conditionals : $P(x | y), P(y | x)$

Marginals : $P(x) = \sum_y P(x, y), P(y) = \sum_x P(x, y)$

Conditionals : $P(x | y) = \frac{P(x, y)}{P(y)}, P(y) > 0$

$P(y | x) = \frac{P(x, y)}{P(x)}, P(x) > 0.$

2.2. BIVARIATE DISTRIBUTION : CONTINUOUS RANDOM VARIABLES

The joint, marginals and conditionals for continuous random variables can be explained below.

1. Joint Probability Density Function

The joint probability density function of the random variables X and Y is denoted by $f_{XY}(x, y)$ or $f(x, y)$ and it should satisfy the following conditions. (AU 2017)

$$(i) f(x, y) \geq 0 \quad \forall (x, y)$$

$$(ii) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$$

(iii) Probability of an event lies in the interval of x in (a, b) and y in (c, d) is

$$P(a < X < b, c < Y < d) = \int_a^b \int_c^d f(x, y) dx dy.$$

2. Marginal Probability Density Functions

(AU 2017)

There are two marginal probability density functions.

1. Marginal Probability Density Function of X

The marginal probability density function of X is denoted by $f_X(x)$ or $f(x)$ and is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy.$$

Properties of marginal probability density function of X :

$$(i) f(x) \geq 0 \quad \forall x.$$

$$(ii) \int_{-\infty}^{\infty} f(x) dx = 1.$$

2. Marginal Probability Density Function of Y

The marginal probability density function of Y is denoted by $f_Y(y)$ or $f(y)$ and is defined as

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

Properties of marginal probability density function of Y :

$$(i) f(y) \geq 0 \quad \forall y$$

$$(ii) \int_{-\infty}^{\infty} f(y) dy = 1$$

3. Conditional Probability Density Functions

There are two conditional probability density functions.

1. Conditional Probability Density Function of Y given X

The conditional probability density function of Y given X is denoted by $f_{Y|X}(y|x)$ or $f(y|x)$ and is defined as

$$f(y|x) = \frac{f(x, y)}{f(x)}, \quad f(x) > 0$$

Properties of conditional probability density function of Y given X :

$$(i) \quad f(y|x) \geq 0 \quad \forall y$$

$$(ii) \quad \int_{-\infty}^{\infty} f(y|x) dy = 1$$

2. Conditional Probability Density Function of X given Y

The conditional probability density function of X given Y is denoted by $f_{X|Y}(x|y)$ or $f(x|y)$ and is defined as

$$f(x|y) = \frac{f(x, y)}{f(y)}, \quad f(y) > 0$$

Properties of conditional probability density function of X given Y :

$$(i) \quad f(x|y) \geq 0 \quad \forall x$$

$$(ii) \quad \int_{-\infty}^{\infty} f(x|y) dx = 1$$

2.3. DISTRIBUTION FUNCTIONS OF BIVARIATE RANDOM VARIABLES

(AU 2018)

1. Joint Distribution Function

The joint distribution function of X and Y is denoted by $F_{XY}(x, y)$ or $F(x, y)$ and is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

For discrete,

$$F(x, y) = \sum_{-\infty}^x \sum_{-\infty}^y P(X = x, Y = y)$$

For continuous,

$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y f(x, y) dx dy.$$

Properties of Joint Distribution Function

1. If a_1, a_2, b_1, b_2 are real numbers, then

$$P(a_1 < X \leq b_1, a_2 < Y \leq b_2) = F_{XY}(b_1, b_2) + F_{XY}(a_1, a_2) - F_{XY}(a_1, b_2) - F_{XY}(b_1, a_2)$$
2. Limits of joint distribution function is 0 and 1, i.e. $0 \leq F(x, y) \leq 1$.
3. $F(-\infty, -\infty) = 0, \quad F(\infty, \infty) = 1$
 $F(-\infty, y) = 0, \quad F(x, -\infty) = 0$
4. If $f(x, y)$ is the joint p.d.f. of (X, Y) , then

$$f(x, y) = \frac{\partial^2 F(x, y)}{\partial x \partial y}$$

2. Marginal Distribution Functions

1. Marginal Distribution Function of X

The marginal distribution function of X is denoted by $F_X(x)$ and is defined as

$$\begin{aligned} F_X(x) &= P(X \leq x) = P(X \leq x, Y < \infty) \\ &= \sum_y P(X \leq x, Y = y) = F(x, \infty) \end{aligned}$$

Properties of marginal distribution function of X :

1. $P(a < X \leq b) = F(b) - F(a)$
2. $0 \leq F(x) \leq 1$
3. $F(-\infty) = 0, F(\infty) = 1$.

2. Marginal Distribution Function of Y

The marginal distribution function of Y is denoted by $F_Y(y)$ or $F(y)$ and is defined as

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X < \infty, Y \leq y) \\ &= \sum_x P(X = x, Y \leq y) = F(\infty, y) \end{aligned}$$

Properties of marginal distribution function of Y :

1. $P(c < Y \leq d) = F(d) - F(c)$
2. $0 \leq F(y) \leq 1$
3. $F(-\infty) = 0, F(\infty) = 1$.

3. Conditional Distribution Functions

1. Conditional Distribution Function of Y given X

The conditional distribution function of Y given X is denoted by $F_{Y|X}(y|x)$ or $F(y|x)$ and is defined as

$$F_{Y|X}(y|x) = P(Y \leq y | X = x)$$

Properties of conditional distribution function of Y given X :

1. $P(c < Y \leq d | X) = F(d|x) - F(c|x)$

2. $0 \leq F(y|x) \leq 1$.
3. $F(-\infty|x) = 0, F(\infty|x) = 1$.

2. Conditional Distribution Function of X given Y

The conditional distribution function of X given Y is denoted by $F_{X|Y}(x|y)$ or $F(x|y)$ and is defined as

$$F_{X|Y}(x|y) = P(X \leq x | Y = y)$$

Properties of conditional distribution function of X given Y :

1. $P(a < X \leq b | y) = F(b|y) - F(a|y)$
2. $0 \leq F(x|y) \leq 1$
3. $F(-\infty|y) = 0, F(\infty|y) = 1$.

2.4. INDEPENDENCE OF RANDOM VARIABLES

Let us consider X, Y be two random variables with the joint p.d.f. $P(x, y)$ or $f(x, y)$ and marginals $P(x), P(y)$ or $f(x), f(y)$ and conditionals $P(x|y), P(y|x)$ or $f(x|y), f(y|x)$ respectively for discrete and continuous random variables. Then the random variables X and Y are said to be stochastically independent or independent if

$$\begin{aligned} \text{and } \left. \begin{aligned} P(y) &= P(y|x) \\ P(x) &= P(x|y) \end{aligned} \right\} \text{for discrete} \\ \text{and } \left. \begin{aligned} f(y) &= f(y|x) \\ f(x) &= f(x|y) \end{aligned} \right\} \text{for continuous} \end{aligned}$$

This can be proved in a simple way. Let X and Y be continuous random variables and if $f(y|x)$ does not depend on x, then

$$\begin{aligned} f(y) &= \int_{-\infty}^{\infty} f(x, y) dx = \int_{-\infty}^{\infty} f(x) \cdot f(y|x) dx \\ &= f(y|x) \cdot \int_{-\infty}^{\infty} f(x) dx \quad (\because f(y|x) \text{ does not depend on } x) \\ &= f(y|x) \quad \left(\because \int_{-\infty}^{\infty} f(x) dx = 1 \right) \end{aligned}$$

$\therefore f(y) = f(y|x)$ provided $f(y|x)$ does not depend on x, i.e. if X and Y are independent.

\therefore The joint p.d.f. can be expressed as

$$f(x, y) = f(x) \cdot f(y|x) = f(x) \cdot f(y) \quad (\text{for continuous})$$

Similarly for discrete $P(x, y) = P(x) \cdot P(y)$.

This leads the definitions of independent random variables.

Definition. Two random variables X and Y are said to be independent if and only if

$$P(x, y) = P(x) \cdot P(y), \text{ for discrete}$$

$$f(x, y) = f(x) \cdot f(y), \text{ for continuous.}$$

Note : This definition is also true for distribution functions, *i.e.* two random variables X and Y are said to be independent if and only if

$$F(x, y) = F(x) \cdot F(y)$$

where $F(x, y)$: Joint distribution function

$F(x)$, $F(y)$ are marginal distribution functions of X and Y .

Marginal distributions

1. $(x, P(x))$ is called marginal distribution of X .
2. $(y, P(y))$ is called marginal distribution of Y .