CHAPTER

3

Mathematical Expectation

3.1. DEFINITION

(AU 2018)

Let X be a random variable, then its mathematical expectation is denoted by E(X) and is defined as

$$E(X) = \begin{cases} \sum x \cdot P(x), & \text{for discrete random variable} \\ \int_{-\infty}^{\infty} x f(x)\, dx, & \text{for continuous random variable.} \end{cases}$$

Expectation exists provided the series/integral is absolutely convergent *i.e.*,

$$\sum |x| \cdot P(x) < \infty, \int_{-\infty}^{\infty} |x| f(x)\, dx < \infty.$$

3.2. EXPECTATION OF FUNCTION OF A RANDOM VARIABLE

If $g(X)$ is a function of a random variable, then expected value of $g(X)$ is defined as

$$E[g(X)] = \begin{cases} \sum g(x) \cdot P(x), & \text{for discrete} \\ \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx, & \text{for continuous} \end{cases}$$

3.3. MOMENTS USING EXPECTATIONS

1. **Non Central moments**

If $g(x) = x^r$, then

$$E(X^r) = \sum x^r \cdot P(x), \quad \text{for discrete}$$

$$= \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx, \quad \text{for continuous}$$

This is rth moment about origin (non-central moments).

If $r = 1$, $E(X) = \text{mean} = \mu_1'$

If $r = 2$, $E(X^2) = \mu_2'$ and $\mu_2 = \mu_2' - \mu_1'^2$

Variance $= E(X^2) - [E(X)]^2$

This is variance of X.

2. Central moments

If $g(X) = (X - \text{mean})^r = (X - E(X))^r = (X - \bar{X})^r$, then

$$E((X - \bar{X})^r) = \sum (x - \bar{x})^r P(x), \quad \text{for discrete}$$

$$= \int_{-\infty}^{\infty} (x - \bar{x})^r . f(x)\, dx, \quad \text{for continuous}$$

This is rth moment about mean or central moments.

If $r = 2$,

$$E(X - \bar{X})^2 = \mu_2 = \text{Variance}$$

If $r = 3$,

$$E(X - \bar{X})^3 = \mu_3 \text{ is third central moment.}$$

Result : $E(C) = C$, C is constant.

If $g(X) = C$, then

$$E(C) = \int_{-\infty}^{\infty} C . f(x)\, dx \quad \text{[let X be a continuous random variable]}$$

$$= C . \int_{-\infty}^{\infty} f(x)\, dx$$

$$= C . 1 \quad (\because \text{ By the definition of p.d.f.})$$

$$= C$$

$\therefore$ $E(C) = C$

In similar way, we can prove for discrete random variable also.

Statistical averages using expectations

1. Mean $= \mu_1' = E(X)$.
2. Variance $= \mu_2 = E[X - E(X)]^2 = E(X^2) - [E(X)]^2$
3. rth moment about origin or non-central moments $\mu_r' = E(X^r)$
4. Central moments (or) rth moment about mean

   $\mu_r = E\{[X - E(X)]^r\}$

3.4. COVARIANCE USING EXPECTATIONS

If X and Y are two random variables, then covariance between the variables X and Y is defined as

$$\text{Cov}(X, Y) = E\{[X - E(X)][Y - E(Y)]\}$$

This formulae also can be expressed as

$$\begin{aligned}\text{Cov}(X, Y) &= E\{XY - XE(Y) - YE(X) + E(X) \cdot E(Y)\}\\ &= E(XY) - E(X) \cdot E(Y) - E(Y) \cdot E(X) + E(X)\,E(Y)\\ &= E(XY) - E(X) \cdot E(Y).\end{aligned}$$

Independence of Random Variables using Expectations

The random variables X and Y are said to be independent if $E(XY) = E(X) \cdot E(Y)$.

i.e. $\quad \text{Cov}(X, Y) = E(XY) - E(X) \cdot E(Y) = 0$

3.5. ADDITION THEOREM OF EXPECTATIONS (AU 2017)

Statement. If X and Y are two random variables, then,

$$E(X + Y) = E(X) + E(Y).$$

Proof : Let X and Y be continuous random variables with joint p.d.f. $f(x, y)$ and marginal p.d.f.s $f(x)$ and $f(y)$ respectively.

[We can prove this theorem for discrete random variables by using summation instead of integration]

By the definition of expectations,

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

Now consider

$$\begin{aligned}E(X + Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x + y)\, f(x, y)\, dx\, dy.\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x, y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x, y)\, dx\, dy\\ &= \int_{-\infty}^{\infty} x \left\{\int_{-\infty}^{\infty} f(x, y)\, dy\right\} dx + \int_{-\infty}^{\infty} y \cdot \left\{\int_{-\infty}^{\infty} f(x, y)\, dx\right\} dy\\ &= \int_{-\infty}^{\infty} x\, f(x)\, dx + \int_{-\infty}^{\infty} y \cdot f(y)\, dy = E(X) + E(Y)\end{aligned}$$

$$E(X + Y) = E(X) + E(Y).$$

Generalization of Addition Theorem of Expectations

Statement. The mathematical expectation of sum of n random variables is equal the sum of their expectations, provided all the expectations exist.

i.e. if $X_1, X_2, \dots X_n$ are 'n' random variables, then

$$E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n)$$

i.e. $$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

Proof : For two random variables X_1 and X_2, we have

$$E(X_1 + X_2) = E(X_1) + E(X_2) \quad \dots(1)$$

$\therefore$ The theorem is true for $n = 2$.

Let us consider the theorem is true for $n = r$,

$$E\left(\sum_{i=1}^{r} X_i\right) = \sum_{i=1}^{r} E(X_i) \quad \dots(2)$$

Consider for $n = r + 1$

$$E\left(\sum_{i=1}^{r+1} X_i\right) = E\left[\sum_{i=1}^{r} X_i + X_{r+1}\right]$$

$$= E\left(\sum_{i=1}^{r} X_i\right) + E(X_{r+1}) \quad [\because \text{ from (1)}]$$

$$= \sum_{i=1}^{r} E(X_i) + E(X_{r+1}) \quad [\because \text{ from (2)}]$$

$$= \sum_{i=1}^{r+1} E(X_i)$$

$$\therefore \quad E\left(\sum_{i=1}^{r+1} X_i\right) = \sum_{i=1}^{r+1} E(X_i)$$

$\therefore$ The theorem is true for $n = r + 1$. Hence by mathematical induction the theorem is true for all values of n.

3.6. MULTIPLICATION THEOREM OF EXPECTATIONS

Statement. If X and Y are independent random variables, then

$$E(XY) = E(X) \,.\, E(Y)$$

Proof : Let X and Y be independent continuous random variables, then by the definition of expectations

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy$$

where $f(x), f(y)$ are marginal p.d.f.s of X and Y.

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\, dx\, dy$$

where $f(x, y)$ is joint p.d.f.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \,.\, f(x) \,.\, f(y) \,.\, dx\, dy.$$

($\because$ X and Y are independent, then $f(x, y) = f(x) \,.\, f(y)$)

$$= \left\{\int_{-\infty}^{\infty} x\, f(x)\; dx\right\}\left\{\int_{-\infty}^{\infty} y\, f(y)\, dy\right\} = E(X) \,.\, E(Y)$$

$$\therefore \qquad E(XY) = E(X) \,.\, E(Y).$$

Generalization of Multiplication Theorem of Expectations

Statement. The mathematical expectation of the product of 'n' independent random variables is equal to the product of their expectations, *i.e.* if $X_1, X_2, \ldots X_n$ are 'n' independent random variables, then symbolically

$$E(X_1 X_2 \ldots X_n) = E(X_1) \,.\, E(X_2) \ldots E(X_n)$$

or $$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

Proof : For two independent random variables X_1 and X_2, we have

$$E(X_1 X_2) = E(X_1) \,.\, E(X_2) \qquad \ldots(1)$$

Let us suppose that the theorem is true for $n = r$

$$\therefore \qquad E\left(\prod_{i=1}^{r} X_i\right) = \prod_{i=1}^{r} E(X_i) \qquad \ldots(2)$$

Consider for $n = r + 1$

$$E\left(\prod_{i=1}^{r+1} X_i\right) = E\left[\left(\prod_{i=1}^{r} X_i\right)(X_{r+1})\right]$$

$$= E\left(\prod_{i=1}^{r} X_i\right) \,.\, E(X_{r+1}) \qquad [\because \text{ from equation (1)}]$$

$$= \prod_{i=1}^{r} E(X_i) \cdot E(X_{r+1}) \quad [\because \text{ from equation (2)}]$$

$$= \prod_{i=1}^{r+1} E(X_i)$$

$$\therefore \quad E\left(\prod_{i=1}^{r+1} X_i\right) = \prod_{i=1}^{r+1} E(X_i)$$

$\therefore$ The theorem is true for $n = r + 1$, hence by mathematical induction, the theorem is true for all values of n.

3.7. PROPERTIES OF EXPECTATIONS

1. Show that E(C) = C

Proof : Let X be continuous random variable with p.d.f. $f(x)$

$$E(C) = \int_{-\infty}^{\infty} C \cdot f(x)\,dx = C \cdot \int_{-\infty}^{\infty} f(x)\,dx = C \cdot 1 = C.$$

$$\therefore \quad E(C) = C$$

2. Show that $E(aX) = aE(X)$.

Proof : Let X be a continuous random variable,

$$E(aX) = \int_{-\infty}^{\infty} ax\, f(x)\,dx = a \int_{-\infty}^{\infty} x\, f(x)\,dx = a \cdot E(X)$$

3. Show that $E(aX + b) = aE(X) + b$ **(AU 2019)**

Proof : Let X be a continuous random variable,

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b)\, f(x)\,dx = a \int_{-\infty}^{\infty} x\, f(x)\,dx + b \int_{-\infty}^{\infty} . f(x)\,dx$$

$$= aE(X) + b.1 = aE(X) + b$$

4. If X and Y are two random variables, then show that $E(aX + bY) = aE(X) + bE(Y)$

Proof : Let X and Y be continuous random variables, $f(x), f(y)$ are p.d.f. of X and Y.

$$E(aX + bY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax + by)\, f(x, y)\,dx\,dy$$

where $f(x, y)$ is the joint p.d.f. of X and Y

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ax\, f(x, y)\,dx\,dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} by\, f(x, y)\,dx\,dy$$

$$= a \int_{-\infty}^{\infty} x \left\{ \int_{-\infty}^{\infty} f(x, y)\, dy \right\} dx + b \int_{-\infty}^{\infty} y \left\{ \int_{-\infty}^{\infty} f(x, y)\, dx \right\} dy$$

$$= a \int_{-\infty}^{\infty} x \,.\, f(x)\, dx + b \,.\, \int_{-\infty}^{\infty} y \,.\, f(y)\, dy = a\,\mathrm{E(X)} + b\,\mathrm{E(Y)}.$$

5. If X is a random variable and a is constant, then show that
(*i*) $E[a\,\psi(X)] = a\,E[\psi(X)]$ (*ii*) $E[\psi(X) + a] = E[\psi(X)] + a$.
where $\psi(X)$ is a function of X.

Proof : Let X be a continuous random variable.

(*i*) $$E[a\psi(X)] = \int_{-\infty}^{\infty} a\,\psi(x)\, f(x)\, dx = a \int_{-\infty}^{\infty} \psi(x)\, f(x)\, dx = a\,E[\psi(X)]$$

(*ii*) $$E(\psi(X) + a) = \int_{-\infty}^{\infty} (\psi(x) + a)\, f(x)\, dx = \int_{-\infty}^{\infty} \psi(x) \,.\, f(x)\, dx + a \,.\, \int_{-\infty}^{\infty} f(x)\, dx$$

$$= E[\psi(X)] + a \qquad \left(\because \int_{-\infty}^{\infty} f(x)\, dx = 1 \right)$$

6. Show that if $X \geq 0$, then $E(X) \geq 0$.

Proof : If X is a continuous random variable, $X \geq 0$ then

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{0}^{\infty} x\, f(x)\, dx \geq 0 \qquad (\text{Since } x \geq 0)$$

(Since $f(x) \geq 0$ always)

$\therefore \quad E(X) \geq 0 \quad \text{for } X \geq 0$

7. If X and Y are two random variables, if $X \leq Y$, then show that $E(X) \leq E(Y)$.

Proof : Since $X \leq Y$

$X - Y \leq 0$

Consider mathematical expectations,

$E(X - Y) \leq 0$

$E(X) - E(Y) \leq 0$

$E(X) \leq E(Y)$

8. Show that $|\,E(X)\,| \leq E\,|\,X\,|$

Proof : Since $X \leq |\,X\,|$

$E(X) \leq E\,|\,X\,|$ ...(1)

and also, since $-X \leq |\,X\,|$

$E(-X) \leq E\,|\,X\,|$

$-E(X) \leq E\,|\,X\,|$ ...(2)

From equations (1) and (2), we get

$|\,E(X)\,| \leq E\,|\,X\,|$.

3.8. PROPERTIES OF VARIANCE

1. Show that Var (C) = 0, C is a constant

Proof : $\text{Var}(C) = E[C - E(C)]^2 = E[C - C]^2$ $(\because E(C) = C)$

$= 0$

2. Show that $\text{Var}(aX) = a^2 \text{Var}(X)$, where a is constant. **(AU 2019)**

Proof :
$$\begin{aligned}\text{Var}(aX) &= E[aX - E(aX)]^2 = E[aX - aE(X)]^2 \\ &= E[a[X - E(X)]]^2 = a^2 E[X - E(X)]^2 \\ &= a^2 \text{Var}(X)\end{aligned}$$

3. Show that $\text{Var}(aX + b) = a^2 \text{Var}(X)$, where a, b are constants.

Proof :
$$\begin{aligned}\text{Var}(aX + b) &= E[aX + b - E(aX + b)]^2 = E(aX + b - aE(X) - b]^2 \\ &= E[a[X - E(X)]^2 = a^2 E[X - E(X)]^2 = a^2 \text{Var}(X).\end{aligned}$$

4. Show that $\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{ Cov}(X, Y)$ where a, b are constants.

Proof :
$$\begin{aligned}\text{Var}(aX + bY) &= E[aX + bY - E(aX + bY)]^2 \\ &= E[aX + bY - aE(X) - bE(Y)]^2 \\ &= E[a(X - E(X)] + b[Y - E(Y)]]^2 \\ &= a^2 E[X - E(X)]^2 + b^2 E[Y - E(Y)]^2 + 2ab\, E\{[X - E(X)][Y - E(Y)]\} \\ &= a^2 \text{Var}(X) + b^2 \text{Var}(Y) + 2ab \text{ Cov}(X, Y).\end{aligned}$$

5. $\text{Var}(aX - bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y) - 2ab \text{ Cov}(X, Y)$ (Proof is left to reader)

6. Show that variance is independent of change of origin but not scale.

Proof : Let $U = \dfrac{X - a}{h}$

where a indicate change of origin

h indicate change of scale.

$$U = \frac{X - a}{h} \Rightarrow X = a + hU$$

Consider variance

$$\begin{aligned}\text{Var}(X) &= \text{Var}(a + hU) = E[a + hU - E(a + hU)]^2 \\ &= E[a + hU - [E(a) + hE(U)]]^2 = E[a + hU - a - hE(U)]^2 \\ &= E[h[U - E(U)]]^2 = h^2 E[U - E(U)]^2\end{aligned}$$

$$\text{Var}(X) = h^2 \text{Var}(U)$$

$\therefore$ Variance is independent of change of origin but not on scale.

3.9. PROPERTIES OF COVARIANCE

1. Show that $\text{Cov}(aX, bY) = ab \text{ Cov}(X, Y)$

Proof :
$$\begin{aligned}\text{Cov}(aX, bY) &= E\{[aX - E(aX)][bY - E(bY)]\} \\ &= E\{[aX - aE(X)][bY - bE(Y)]\}\end{aligned}$$

$= E\{a\,[X - E(X)]\; b\,[Y - E(Y)\}$

$= ab\; E\{X - E(X)]\,[Y - E(Y)]\} = ab\; Cov\,(X, Y).$

2. Show that

Cov $(X + a, Y + b) =$ Cov (X, Y)

Proof :

$$\begin{aligned} Cov\,(X + a, Y + b) &= E\,\{[X + a - E(X + a)]\,[Y + b - E(Y + b)]\} \\ &= E\{[X + a - E(X) - a]\,[Y + b - E(Y) - b]\} \qquad (\because E(a) = a, E(b) = b) \\ &= E\{[X - E(X)]\,[Y - E(Y)]\} = Cov\,(X, Y). \end{aligned}$$

3. Show that Cov $(aX + b, cY + d) = ac$ Cov (X, Y)

Proof :

$$\begin{aligned} Cov\,(aX + b, cY + d) &= E\{[aX + b - E(aX + b)]\,[cY + d - E(cY + d)]\} \\ &= E\{[aX + b - aE(X) - b]\,[cY + d - cE(Y) - d]\} \\ &= E\{a[X - E(X)]\,c\,[Y - E(Y)]\} = ac\; E\{(X - E(X)]\,[Y - E(Y)]\} \\ &= ac\; Cov\,(X, Y). \end{aligned}$$

4. Show that Cov $(aX + bY, cX + dY) = ac$ Var $(X) + bd$ Var $(Y) + (ad + bc)$ Cov (X, Y)

Proof :

$$\begin{aligned} Cov\,(aX + bY, cX + dY) &= E\{[aX + bY - E(aX + bY)]\,[cX + dY - E(cX + dY)]\} \\ &= E\{[a\,[X - E(X)] + b[Y - E(Y)]]\,[c[X - E(X)] + d[Y - E(Y)]]\} \\ &= ac\; E\{[X - E(X)]^2\} + bc\; E\{X - E(X)]\,[Y - E(Y)]\} \\ &\qquad + ad\; E\{[X - E(X)]\,[Y - E(Y)]\} + bd\; E\{[Y - E(Y)]^2\} \\ &= ac\; Var\,(X) + bc\; Cov\,(X, Y) + ad\; Cov\,(X, Y) + bd\; Var\,(Y) \\ &= ac\; Var\,(X) + bd\; Var\,(Y) + (ad + bc)\; Cov\,(X, Y) \end{aligned}$$

5. Prove that covariance is independent of change of origin but not scale.

Proof : Let $U = \dfrac{X - a}{h}$, $V = \dfrac{Y - b}{K}$

where a, b indicate change of origin

h, K indicate change of scale.

$$U = \frac{X - a}{h}, \quad V = \frac{Y - b}{K}$$

$\Rightarrow \quad X = a + hU, \quad Y = b + KV$

Consider covariance of X and Y.

$$\begin{aligned} Cov\,(X, Y) &= E\{[X - E(X)]\,[Y - E(Y)]\} \\ &= E\{[a + hU - E(a + hU)]\,[b + KV - E(b + KV)]\} \\ &= E\{[a + hU - a - hE(U)]\,[b + KV - b - KE(V)]\} \\ &= E\{[h\,[U - E(U)]\,K[V - E(V)]\} = hK\; E\{[U - E(U)]\,[V - E(V)]\} \\ &= hK\; Cov\,(U, V) \end{aligned}$$

$\therefore$ Covariance is independent of change of origin but not scale.

3.10. CAUCHY–SCHWARTZ INEQUALITY (AU 2019)

Statement. If X and Y are random variables, then $[E(XY)]^2 \le E(X^2) \cdot E(Y^2)$

Proof : Let us consider a real valued function on t, defined as

$$z(t) = E(X + tY)^2$$

for all real X, Y and t, $(X + tY)^2 \ge 0$

then $$z(t) = E[(X + tY)^2] \ge 0 \quad \forall\ t$$

$$\Rightarrow \quad E(X^2) + t^2 E(Y^2) + 2t\, E(XY) \ge 0$$

$z(t)$ is a quadratic expression in t

Let $$\phi(t) = At^2 + Bt + C \ge 0$$

where $$A = E(Y^2),\ B = 2E(XY),\ C = E(X^2)$$

The quadratic expression $\phi(t) \ge 0$ possible only when the discriminant $B^2 - 4AC \le 0$.

$$\therefore \quad B^2 - 4AC \le 0$$

$$[2E(XY)]^2 - 4E(Y^2) \cdot E(X^2) \le 0$$

$$\Rightarrow \quad [E(XY)]^2 - E(X^2) \cdot E(Y^2) \le 0$$

$$[E(XY)]^2 \le E(X^2) \cdot E(Y^2)$$

3.11. CHEBYCHEV'S INEQUALITY (AU 2017, 2019)

Statement. If X is a random variable with the mean μ and variance σ^2, then for any positive number K, we have

$$P\{|X - \mu| \ge K\sigma\} \le \frac{1}{K^2}$$

(or) $$P\{|X - \mu| < K\sigma\} \ge 1 - \frac{1}{K^2}$$

Proof : Let X be a continuous random variable.

[For a discrete random variable, change integration by summation].

By the definition of variance

$$\text{Var}(X) = \sigma_X^2 = E[X - E(X)]^2 = E[X - \mu]^2$$

$$= \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx \quad \text{(for a continuous random variable)}$$

$$= \int_{-\infty}^{\mu - K\sigma} (x - \mu)^2 f(x)\, dx + \int_{\mu - K\sigma}^{\mu + K\sigma} (x - \mu)^2 f(x)\, dx + \int_{\mu + K\sigma}^{\infty} (x - \mu)^2 f(x)\, dx$$

$$\ge \int_{-\infty}^{\mu - K\sigma} (x - \mu)^2 f(x)\, dx + \int_{\mu + K\sigma}^{\infty} (x - \mu)^2 f(x)\, dx.$$

$$\left(\because \int_{\mu - K\sigma}^{\mu + K\sigma} (x - \mu)^2 f(x)\, dx \ge 0\right)$$

For the first integral

$$x \le \mu - K\sigma$$

$$x - \mu \le -K\sigma \quad \text{or} \quad -(x-\mu) \ge K\sigma$$

$$\Rightarrow \quad (x-\mu)^2 \ge K^2\sigma^2$$

For the second integral

$$x \ge \mu + K\sigma$$

$$\Rightarrow \quad x - \mu \ge K\sigma \qquad (x-\mu)^2 \ge K^2\sigma^2$$

$\therefore$ It is true for both the integrals

$$\because \quad (x-\mu)^2 \ge K^2\sigma^2$$

$$\therefore \quad \sigma^2 \ge \int_{-\infty}^{\mu - K\sigma} K^2\sigma^2 f(x)\,dx + \int_{\mu + K\sigma}^{\infty} K^2\sigma^2 f(x)\,dx$$

$$\sigma^2 \ge K^2\sigma^2 \left\{ \int_{-\infty}^{\mu - K\sigma} f(x)\,dx + \int_{\mu + K\sigma}^{\infty} f(x)\,dx \right\}$$

$$\sigma^2 \ge K^2\sigma^2 \{P(X < \mu - K\sigma) + P(X > \mu + K\sigma)\}$$

$$\sigma^2 \ge K^2\sigma^2 \{P(X - \mu < -K\sigma) + P(X - \mu > K\sigma)\} \ge K^2\sigma^2 P[|X - \mu| \ge K\sigma]$$

$$\therefore \quad \sigma^2 \ge K^2\sigma^2 P[|X - \mu| \ge K\sigma]$$

$$1 \ge K^2 P[|X - \mu| \ge K\sigma]$$

$$\frac{1}{K^2} \ge P[|X - \mu| \ge K\sigma]$$

$$\Rightarrow \quad P[|X - \mu| \ge K\sigma] \le \frac{1}{K^2}$$

This can also be

$$P[|X - \mu| \le K\sigma] \ge 1 - \frac{1}{K^2}.$$

Result

1. Show that $\beta_2 \ge \beta_1 + 1$

   Proof : To prove this, first we consider the result $\beta_2 \ge \beta_1 - (2K + K^2)$, proof of this result is as follows.

   Consider $E(X) = 0$ without loss of generality.

   Then $\mu_r = E(X^r)$ $\qquad (\because E(X) = 0)$

   Consider a real valued function on t as

$$Z(t) = E[X^2 + tX + K\mu_2]^2 \ge 0 \quad \forall\, t$$

$$\Rightarrow \quad E[X^4 + t^2X^2 + K^2\mu_2{}^2 + 2tX^3 + 2K\mu_2X^2 + 2K\mu_2 tX] \ge 0$$

$$\Rightarrow \quad E(X^4) + t^2E(X^2) + K^2\mu_2{}^2 + 2t\,E(X^3) + 2K\mu_2\,E(X^2) + 2K\mu_2 t\,E(X) \ge 0$$

$$\Rightarrow \quad \mu_4 + t^2\mu_2 + K^2\mu_2{}^2 + 2t\mu_3 + 2K\,\mu_2{}^2 \ge 0 \qquad (\because E(X) = 0)$$

$$\Rightarrow \quad t^2\mu_2 + 2t\mu_3 + \mu_4 + K^2\mu_2{}^2 + 2K\mu_2{}^2 \ge 0$$

Since $z(t)$ is quadratic expression in t and

$z(t) \geq 0 \quad \forall\, t$ is possible only for discriminant ≤ 0.

$$\therefore \quad (2\mu_3)^2 - 4\,\mu_2\,[\mu_4 + K^2\mu_2^2 + 2K\mu_2^2] \leq 0.$$

Divide by $4\mu_2^3$ (>0)

$$\frac{\mu_3^2}{\mu_2^3} - \left[\frac{\mu_4}{\mu_2^2} + K^2 + 2K\right] \leq 0$$

$$\beta_1 - [\beta_2 + K^2 + 2K] \leq 0$$

$$\beta_1 \leq \beta_2 + K^2 + 2K$$

$$\Rightarrow \quad \beta_2 + K^2 + 2K \geq \beta_1$$

$$\Rightarrow \quad \beta_2 \geq \beta_1 - (K^2 + 2K)$$

$\therefore$ Let $K = -1$, we get

$$\beta_2 \geq \beta_1 - (-1)$$

$$\beta_2 \geq \beta_1 + 1$$

Note : 1. If we let $K = 0$, we get $\beta_2 \geq \beta_1$

2. Since $\beta_2 \geq \beta_1 + 1$ then we can show $\beta_2 \geq 1$.